

# LONG HITTING TIME, SLOW DECAY OF CORRELATIONS AND ARITHMETICAL PROPERTIES

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**ABSTRACT.** Let  $\tau_r(x, x_0)$  be the time needed for a point  $x$  to enter for the first time in a ball  $B_r(x_0)$  centered in  $x_0$ , with small radius  $r$ . We construct a class of translations on the two torus having particular arithmetic properties (Liouville components with intertwined denominators of convergents) not satisfying a logarithm law, i.e. such that for typical  $x, x_0$

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \infty.$$

By considering a suitable reparametrization of the flow generated by a suspension of this translation, using a previous construction by Fayad, we show the existence of a mixing system on three torus having the same properties. The speed of mixing of this example must be subpolynomial, because we also show that: in a system having polynomial decay of correlations, the  $\limsup_{r \rightarrow 0}$  of the above ratio of logarithms (which is also called the upper hitting time indicator) is bounded from above by a function of the local dimension and the speed of correlation decay.

More generally, this shows that reparametrizations of torus translations having a Liouville component cannot be polynomially mixing.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let  $(X, T, \mu)$  be an ergodic system on a metric space  $X$  and fix a point  $x_0 \in X$ . For  $\mu$ -almost every  $x \in X$ , the orbit of  $x$  goes closer and closer to  $x_0$  entering (sooner or later) in every positive measure neighborhood of the target point  $x_0$ .

For several applications it is useful to quantify the speed of approaching of the orbit of  $x$  to  $x_0$ . In the literature this has been done in several ways:

- Hitting time (also called waiting time). Let  $B_r(x_0)$  be a ball with radius  $r$  centered in  $x_0$ . We consider the time

$$\tau_r(x, x_0) = \min\{n \in \mathbb{N}^+ : T^n(x) \in B_r(x_0)\}$$

needed for the orbit of  $x$  to enter in  $B_r(x_0)$  for the first time. We consider the asymptotic<sup>1</sup> behavior of  $\tau_r(x, x_0)$  as  $r$  decreases to 0. Often this is a power law of the type  $\tau_r \sim r^{-d}$  and then it is interesting to extract the exponent  $d$  by looking at the behavior of  $\frac{\log \tau_r(x, x_0)}{-\log r}$  for small  $r$ . In many systems (having generic arithmetical properties or fast decay of correlations, see e.g. [1, 13, 15, 21, 20]) this quantity converges to the local dimension  $d_\mu(x_0)$  of  $\mu$  at  $x_0$ . We remark that in these systems

$$(1.1) \quad \tau_r \sim \mu(B_r(x_0))^{-1}.$$

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<sup>1</sup>For two functions of the same variable we intend the asymptotic symbol  $f(x) \sim g(x)$  to mean  $\lim_{x \rightarrow 0} \frac{\log f(x)}{\log g(x)} = 1$ .

- Logarithm (like) law. Let  $d_n(x, x_0) = \min_{1 \leq i \leq n} \text{dist}(T^i(x), x_0)$ . We consider the asymptotic behavior of  $d_n(x, x_0)$  as  $n$  goes to  $\infty$ . In several examples of flows on suitable spaces (mostly constructed by algebraic means and having fast decay of correlations), estimates on the behavior for  $d_n$  (or distances between suitable projections of the points  $x, x_0$ ) are given, in particular when  $x_0$  is the point at infinity (see e.g. [2, 17, 22, 24, 25, 28]). These problems are deeply connected with diophantine approximation and several geometric questions.
- Dynamical Borel-Cantelli Lemma (Shrinking targets). We consider a family of balls  $B_i = B_{r_i}(x_0)$  with  $i \in \mathbb{N}$  centered in  $x_0$  and such that  $r_i \rightarrow 0$  and we ask if  $x \in \limsup_i T^{-i}(B_i)$  or equivalently  $T^i(x) \in B_i$  for infinitely many  $i$  (other families of decreasing sets have also been considered similarly). Here in some class of systems with fast decay of correlations, or generic arithmetical properties (see e.g. [7, 9, 11, 16, 23]) it is possible to obtain results like

$$(1.2) \quad \sum_{i=1}^n \mu(B_i) = \infty \Rightarrow x \in \limsup_i T^{-i}(B_i)$$

for a.e.  $x \in X$ .

Each of the above points has a large related bibliography which cannot be cited exhaustively (see references in cited articles). The above points of view are strictly related. It is easy to see (see Proposition 11) that

$$(1.3) \quad \lim_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \left( \lim_{n \rightarrow \infty} \frac{-\log d_n(x, x_0)}{\log n} \right)^{-1}$$

when limits exist. Hence the hitting time and the logarithm law approaches are somewhat equivalent. A statement like equation (1.2) instead is somewhat slightly stronger (see [16] or remark 3).

Given the abundance of relations between the hitting time and measure of the balls (and hence local dimension), it is worth to look for examples where there is no such relation. Remark that, by what is said above, such a system should not have fast decay of correlations or generic arithmetical properties in some sense. In [21] it is proved that if we consider a rotation  $x \mapsto x + \alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  and  $\alpha$  is a Liouville irrational, then for each  $x_0$  it holds

$$\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \infty,$$

for almost every  $x$ , while on the other hand, for *every* irrational  $\alpha$ ,

$$(1.4) \quad \liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = 1.$$

Remark that 1 is equal to the local dimension for every  $x_0$ , hence in this kind of systems a relation like Equation (1.1) does not hold for the limsup behavior but holds for the liminf one. Trying to generalize this kind of results to the  $d$  dimensional torus, it is natural to ask if there are translations on the  $d$ -torus violating the above equality (1.4) with 1 replaced by  $d$  (indeed this problem was posed in [29], section 5). By (1.3) (more precisely, by Proposition 11) such translations would also violate

the logarithm law

$$(1.5) \quad \limsup_{i \rightarrow \infty} \frac{\log d_i(x, x_0)}{-\log i} = \frac{1}{d}.$$

We remark that by Khintchin-Groshev Theorem (see e.g. [29], section 5 for a statement from our point of view) this equation must be satisfied for almost every translation on the  $d$ -torus.

In section 4 we construct translations on the torus  $\mathbb{T}^2$  where logarithm law (1.5) does not hold; this requires particular arithmetic properties. More precisely (see Theorem 3):

**Theorem A.** *Let  $T_{(\alpha, \alpha')}$  be the translation on the torus  $\mathbb{T}^2$  by vector  $(\alpha, \alpha') \in \mathbb{R}^2$ . If  $\alpha, \alpha'$  are irrationals whose continued fraction expansions have denominators of convergents  $q_n, q'_n$  which, for some  $\gamma > 1$ , satisfy (eventually)*

$$q'_n \geq q_n^\gamma ; \quad q_{n+1} \geq q_n'^\gamma$$

*then for each  $x_0$*

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} \geq \gamma$$

*holds for almost each  $x \in \mathbb{T}^2$ .*

This answers negatively the above mentioned problem of [29] (see also QUESTION 3 in [30]).

In [11] Fayad provides an example of a mixing system which does not have the monotone shrinking target property. In such system relation (1.2) is not satisfied for some decreasing sequence of balls. The example is given by a reparametrization of a translation flow having arithmetical properties which are included in the class considered above. Hitting time indicators are well behaved under bounded reparametrizations and Poincaré sections (see section 5). By these properties, using Fayad's construction and Theorem A, we can strengthen the result proved in [11], by the following (see Corollary 1)

**Theorem B.** *There exists a smooth, mixing system  $(\mathbb{T}^3, T, \mu)$  on the three dimensional torus, with absolutely continuous invariant measure  $\mu$  (and continuous positive density) such that for every  $x_0$*

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \infty$$

*holds for  $\mu$  almost every  $x \in \mathbb{T}^3$ .*

We remark that this is slightly stronger than what is proved in [11] because, as said above, the shrinking target property implies logarithm law. In this example the speed of correlation decay (see Definition 6) must be lower than any power law.

This fact follows from the general result proved in the last section of the paper: there is a relation between local dimension, decay of correlations and hitting time. Several results proving a relation between hitting time and ball's measure need some sort of rapid decay of correlations. Polynomial decay is enough to prove an upper bound for the hitting time. We state this in the following (for a more general result see Theorem 5):

**Theorem C.** *If a system on a manifold of dimension  $d$  has absolutely continuous invariant probability measure with continuous and strictly positive density (the assumption on the measure can be largely relaxed) and polynomial decay of correlations (on Lipschitz observables) with exponent  $p$ , then for each  $x_0$*

$$d \leq \limsup_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} \leq d + \frac{2d+2}{p}$$

*holds for  $\mu$ -almost each  $x$ .*

This theorem and invariance of hitting time under positive speed reparametrizations (Proposition 9) give a more general upper bound on the decay of correlations for reparametrizations of torus translations, depending on its arithmetical properties (see Corollary 3):

**Theorem D.** *If  $(\mathbb{T}^d, T)$  is the time-1 map of any reparametrization with positive speed of an irrational translation flow with direction  $\alpha = (\alpha_1, \dots, \alpha_d)$  and some of the  $\alpha_i$  has type  $\gamma > d$  (see section 4 for definition of type) then the speed of decay of correlations of this system is slower than a power law with exponent  $\frac{2d+2}{\gamma-d}$ .*

In particular a reparametrization of a translation with an angle having infinite type must have *subpolynomial* decay of correlations. We remark (see [12]) that there are polynomially mixing reparametrizations with finite type angles (and no strictly positive speed).

This paper is organized as follows: in section 2 we give basic definitions and state some general facts about hitting time, local dimension and relations between them. In section 3 we recall basic facts on continued fractions and review known results about type and circle rotations. In section 4 we prove Theorem A (Theorem 3). In section 5 we state some easy relations about hitting time in flows and its behavior under reparametrizations, time one map, and poincare sections, allowing to deduce our Theorem B from Theorem A and Fayad's result (see Corollary 1). In section 6 we prove a result on speed of decay of correlation and hitting time (Theorem 5); this will imply Theorem C and Theorem D. Theorem A and Theorem D together imply that Fayad's example has subpolynomial decay of correlations.

In the Appendix we give some auxiliary proposition which are useful to relate some of the different point of views on the subject mentioned at the beginning of this introduction.

**Acknowledgements.** We would like to thank Bassam Fayad, for stimulating discussions and for pointing us some relevant papers. Second author would like to thank SISSA/ISAS (Trieste) where a first part of research work was carried; he would like also to thank his parents for financial support during last part of the work.

## 2. HITTING TIME AND LOCAL DIMENSION: BASIC FACTS

Let  $(X, T)$  be a discrete time dynamical system where  $X$  is a separable metric space equipped with a Borel finite measure  $\mu$  and  $T : X \rightarrow X$  is a measurable map.

**Definition 1.** *The first entrance time of the orbit of  $x$  in the ball  $B(x_0, r)$  with center  $x_0$  and radius  $r$  is*

$$\tau_r(x, x_0) = \min\{n \in \mathbb{N}^+ : T^n(x) \in B(x_0, r)\}.$$

We want to study the behaviour of  $\tau_r(x, x_0)$  as  $r \rightarrow 0$ . In many interesting cases this is a power law  $\tau_r(x, x_0) \sim r^{-R}$ . In order to extract the exponent, we define

**Definition 2.** *The upper and lower hitting time indicators are*

$$(2.1) \quad \overline{R}(x, x_0) = \limsup_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r}, \quad \underline{R}(x, x_0) = \liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r}.$$

If for some  $r$ ,  $\tau_r(x, x_0)$  is not defined then  $\overline{R}(x, x_0)$  and  $\underline{R}(x, x_0)$  are set to be equal to infinity. We remark that the indicators  $\overline{R}(x)$  and  $\underline{R}(x)$  of quantitative recurrence defined in [3] are obtained as a special case,  $\overline{R}(x) = \overline{R}(x, x)$ ,  $\underline{R}(x) = \underline{R}(x, x)$ .

We recall some basic properties of  $R(x, x_0)$  which follow from the definition:

**Proposition 1.**  *$\overline{R}(x, x_0)$  satisfies the following properties (and the same is true with  $\overline{R}$  replaced by  $\underline{R}$ ):*

- $x \notin T^{-1}(x_0)$  implies  $\overline{R}(x, x_0) = \overline{R}(T(x), x_0)$ .
- If  $F$  is bilipschitz and

$$(2.2) \quad \begin{array}{ccc} X & \xrightarrow{F} & Y \\ \downarrow T_1 & & \downarrow T_2 \\ X & \xrightarrow{F} & Y \end{array}$$

*commutes, then  $\overline{R}(x, x_0) = \overline{R}(F(x), F(x_0))$ .*

- *If  $T$  is bilipschitz, and  $x \notin T^{-1}(T(x_0))$  then  $\overline{R}(x, x_0) = \overline{R}(x, T(x_0))$ .*

By the above properties, if  $T$  is an ergodic isometry on the  $d$ -torus then  $\overline{R}(x, x_0) = \overline{R}$ ,  $\underline{R}(x, x_0) = \underline{R}$  are constant for almost every  $x, x_0$ .

The continous limit in the definition of hitting time indicator can be reduced to a discrete limit:

**Lemma 1.** *Let  $r_n$  be a decreasing sequence of reals, such that  $r_n \rightarrow 0$ . Suppose that there is a constant  $1 > c > 0$  satisfying  $r_{n+1} > cr_n$  eventually as  $n$  increases. If  $\tau_r : \mathbb{R} \rightarrow \mathbb{R}$  is decreasing, then  $\liminf_{n \rightarrow \infty} \frac{\log \tau_{r_n}}{-\log r_n} = \liminf_{r \rightarrow 0} \frac{\log \tau_r}{-\log r}$  and  $\limsup_{n \rightarrow \infty} \frac{\log \tau_{r_n}}{-\log r_n} = \limsup_{r \rightarrow 0} \frac{\log \tau_r}{-\log r}$ .*

*Proof.* If  $r_n \geq r \geq r_{n+1} \geq cr_n$  then  $\tau_{r_{n+1}} \geq \tau_r \geq \tau_{r_n}$ , moreover  $\log r_n \geq \log r \geq \log r_{n+1} \geq \log cr_n \geq \log cr_{n+1}$  hence for  $n$  big enough

$$\frac{\log \tau_{r_{n+1}}}{-\log r_{n+1}} \geq \frac{\log \tau_r}{-\log r - \log c} \geq \frac{\log \tau_{r_n}}{-\log r_n - 2 \log c},$$

which gives the statement by taking the limits.  $\square$

We now recall some definitions about local dimension. If  $X$  is a metric space and  $\mu$  is a measure on  $X$  the local dimension of  $\mu$  at  $x_0$  is defined as follows

**Definition 3.** *The upper and lower local dimensions at  $x_0$  are*

$$\overline{d}_\mu(x_0) = \limsup_{r \rightarrow 0} \frac{\log \mu(B_r(x_0))}{\log r}, \quad \underline{d}_\mu(x_0) = \liminf_{r \rightarrow 0} \frac{\log \mu(B_r(x_0))}{\log r}$$

If the limit exists we denote its value as  $d_\mu(x_0) = \lim_{r \rightarrow 0} \frac{\log \mu(B_r(x_0))}{\log r}$ . In this case  $\mu(B_r(x_0)) \sim r^{d_\mu(x_0)}$ . If  $\overline{d}_\mu(x_0) = \underline{d}_\mu(x_0) = d$  almost everywhere the system is called exact dimensional. In this case many notions of dimension of a measure coincide. In particular  $d$  is equal to the infimum Hausdorff dimension of full measure sets:

$d = \inf\{\dim_H Z : \mu(Z) = 1\}$ . This happens in a large class of systems, for example in systems having nonzero Lyapunov exponents almost everywhere (see the book [26], chapter 2).

In general measure preserving systems it is rather easy to prove that behavior of the hitting time is related to the invariant measure of the system.

**Proposition 2.** ([14]) *If  $(X, T, \mu)$  is a dynamical system over a separable metric space, with an invariant measure  $\mu$  (not necessarily finite), then for each  $x_0$*

$$(2.3) \quad \underline{R}(x, x_0) \geq \underline{d}_\mu(x_0) \text{ , } \overline{R}(x, x_0) \geq \overline{d}_\mu(x_0)$$

*holds for  $\mu$ -almost every  $x$ .*

**Remark 1.** *Relations of this type can appear (see e.g. statement of Theorem 5) in the form of a direct logarithm law between waiting time and measure; for example*

$$\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log \mu(B_r(x_0))} \geq 1$$

*implies (but it is slightly more precise of) first inequality in (2.3) while analogously*

$$\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log \mu(B_r(x_0))} \leq 1$$

*implies the upper bound for hitting time:  $\overline{R}(x, x_0) \leq \overline{d}_\mu(x_0)$ . Implications become equivalences when exact-dimensional measure is assumed.*

In systems with decay of correlations (see Defintion 6) faster than any power law the above inequalities become equalities (for precise statements, see [13] or Theorem 5).

### 3. CONTINUED FRACTIONS, TYPE AND CIRCLE ROTATIONS

We briefly recall the basic definitions and properties of continued fractions ( for a general reference see e.g. [18]) that will be needed in the sequel. Let  $\alpha$  be an irrational number, and denote by  $[a_0; a_1, a_2, \dots]$  its continued fraction expansion:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} =: [a_0; a_1, a_2, \dots].$$

The integers  $a_0, a_1, a_2, \dots$  are called partial quotients of  $\alpha$  and are all positive except for  $a_0$ . As usual, we define inductively the sequences  $p_n$  and  $q_n$  by:

$$\begin{aligned} p_{-1} &= 1, & p_0 &= 0, & p_{k+1} &= a_{k+1}p_k + p_{k-1} \text{ for } k \geq 0; \\ q_{-1} &= 0, & q_0 &= 1, & q_{k+1} &= a_{k+1}q_k + q_{k-1} \text{ for } k \geq 0. \end{aligned}$$

The fractions  $p_n/q_n$  are called the *convergents* of  $\alpha$ , as they do in fact converge to it. Moreover they can be seen as *best approximations* of  $\alpha$  in the following sense. Denote by  $\|x\| := \min_{n \in \mathbb{Z}} |x - n|$  the distance of a real number from the integers. Then  $q = q_n$  for some  $n$  if and only if

$$\|q\alpha\| < \|q'\alpha\| \text{ for every positive } q' < q$$

and  $p_n$  is the integer such that  $\|q_n\alpha\| = |q_n\alpha - p_n|$ .

**Proposition 3.** *For any irrational number  $\alpha$  we have that the sequence of its convergents  $q_n$  eventually grows faster than any fixed power of  $n$ .*

*Proof.*  $q_{n+1} \geq q_n + q_{n-1}$  so  $q_n \geq f_n$  where  $f_n$  is Fibonacci sequence.  $\square$

**Proposition 4.** ([18], *Thm. 9 and Thm. 13*)

$$\frac{1}{2} \frac{1}{q_{n+1}} < \frac{1}{q_n + q_{n+1}} < \|q_n \alpha\| < \frac{1}{q_{n+1}}$$

To measure how well an irrational number is approximated by rational numbers one introduces the notion of type.

**Definition 4.** *The type (or Diophantine exponent<sup>2</sup>) of an irrational number  $\alpha$  is defined as*

$$\gamma(\alpha) := \sup\{\beta : \liminf_{q \rightarrow \infty} q^\beta \|q\alpha\| = 0\}$$

In terms of the  $q_n$ s, it is easy to show that this is equal to

$$\limsup_{n \rightarrow \infty} \frac{\log q_{n+1}}{\log q_n}$$

(a proof can be given using Propositions 12, 13 and 4).

Every irrational number has type  $\geq 1$ . The set of number of type 1 (also known as Roth type) is of full measure; the set of numbers of type  $\gamma$  has Hausdorff dimension  $\frac{2}{\gamma+1}$ . There exist numbers of infinite type, called *Liouville* numbers; their set is dense and uncountable and has zero Hausdorff dimension.

We consider a rotation of the circle as a translation on  $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$  by a number  $\alpha \in \mathbb{R}$  and we denote it by  $T_\alpha : x \mapsto x + \alpha \pmod{1}$ . For an irrational rotation the following relations are known between hitting time indicator of  $T_\alpha$  and the type of the rotation number  $\gamma(\alpha)$ .

**Theorem 1.** ([21]) *For a fixed point  $x_0$  and for Lebesgue-almost every  $x$*

$$\overline{R}(x, x_0) = \gamma, \quad \underline{R}(x, x_0) = 1.$$

This result was preceded by an analogous one involving recurrence time.

**Theorem 2.** ([8]) *For every  $x$*

$$(3.1) \quad \underline{R}(x, x) = \frac{1}{\gamma}, \quad \overline{R}(x, x) = 1.$$

Notice that for each translation  $\underline{R}(x, x_0)$  is almost everywhere equal to the dimension of the circle. It is natural to ask if this generalizes to translations on  $n$ -tori. In next section we will show that this does not hold even for  $n = 2$ .

#### 4. LONG HITTING TIME FOR TORUS TRANSLATIONS

We denote by  $T_{(\alpha, \alpha')}$  the translation by vector  $(\alpha, \alpha') \in \mathbb{R}^2$  on the torus  $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ ; we write  $\tau_r^{(\alpha, \alpha')}(x)$  for the hitting time needed for a point  $x \in \mathbb{T}^2$  to reach the ball of radius  $r$  centered in a fixed point  $x_0$ , where the metric is given by sup distance. This transformation is the direct product of two rotations of the circle  $(T_\alpha, T_{\alpha'})$  with hitting time functions  $\tau_r^\alpha(x), \tau_r^{\alpha'}(x)$  as above. The notation for continued fractions is easily guessed (e.g.  $q'_n$  are denominators of convergents of  $\alpha'$ ) and  $\gamma, \gamma'$  are respectively the types of  $\alpha$  and  $\alpha'$ . In this section  $x_0$  is assumed to be 0, for circle case, or  $(0, 0)$ , for the torus case. This can be done without loss

<sup>2</sup>Type plus one is also called irrationality exponent, see for example [27].

of generality regarding statements about hitting time indicator (see Proposition 1)

As in the case of one dimensional rotations, the hitting time behavior of  $T_{(\alpha, \alpha')}$  will depend on the arithmetical properties of  $(\alpha, \alpha')$ . The following is an easy (and not sharp at all) estimation for the limsup indicator

**Proposition 5.** *If  $T_{(\alpha, \alpha')}$  is a translation on the two torus as above and  $\gamma, \gamma'$  are respectively the type of  $\alpha$  and  $\alpha'$ , then for each  $x_0$*

$$\overline{R}(x, x_0) \geq \max(\gamma, \gamma').$$

*Proof.* Since the distance on the torus is the sup distance then  $T_{(\alpha, \alpha')}^n(x)$  is near  $x_0$  only if both coordinates are, so  $\tau_r^{(\alpha, \alpha')}(x, x_0) \geq \max(\tau_r^{(\alpha)}(x, x_0), \tau_r^{(\alpha')}(x, x_0))$  and the statement follows straightforwardly by this.  $\square$

The above proposition can be trivially generalized to the  $n$  torus. This proposition implies that if one of the angles has infinite type then the limsup indicator of the whole translation is infinite.

The key to obtain non trivial lower bound for liminf indicator is to consider irrationals with intertwined denominators of convergents. Take  $\gamma > 1$  and let  $Y_\gamma \subset \mathbf{R}^2$  be the class of couples of irrationals  $(\alpha, \alpha')$  given by the following conditions on their convergents to be satisfied eventually:

$$\begin{aligned} q'_n &\geq q_n^\gamma; \\ q_{n+1} &\geq q_n'^\gamma. \end{aligned}$$

We note that each  $Y_\gamma$  is uncountable and dense in  $[0, 1] \times [0, 1]$  and each irrational of the couple is of type at least  $\gamma^2$ . The set  $Y_\infty = \bigcap_\gamma Y_\gamma$  is also uncountable and dense in unit square and both coordinates of the couple are Liouville numbers (cfr. with construction appearing in Fayad's example, see Theorem 4).

Our main result in this section is the following:

**Theorem 3.** *If  $T_{(\alpha, \alpha')}$  is a translation of the two torus by a vector  $(\alpha, \alpha') \in Y_\gamma$  and  $x_0 \in \mathbb{T}^2$ , then for Lebesgue-almost every  $x \in \mathbb{T}^2$*

$$\underline{R}(x, x_0) \geq \gamma.$$

*In particular, for  $(\alpha, \alpha') \in Y_\infty$  almost everywhere  $\underline{R}(x, x_0) = \infty$ .*

We will prove this long hitting time behaviour reducing to the one-dimensional case: Lemma 2 locates in one dimension the radii where hitting time is not so long, i. e. 'moments' when orbit is near the target; alternating character of the convergents of  $\alpha$  and  $\alpha'$  implies that when one coordinate is near the target the other is far from it and let us deduce long hitting time behaviour for the torus.

For circle rotations the following proposition gives the relation between measure of points with fixed hitting time and convergents of the continued fraction, and will be used to prove our key Lemma. We omit the easy proof (also follows from Proposition 6 of [21]).

**Proposition 6.** *Given  $r$  such that  $\|q_n \alpha\| < 2r \leq \|q_{n-1} \alpha\|$  we have*

$$\begin{aligned} \mu\{x : \tau_r^\alpha(x) = k\} &= 2r & \text{for } k \leq q_n; \\ \mu\{x : \tau_r^\alpha(x) = k\} &\leq \|q_n \alpha\| & \text{for } k > q_n. \end{aligned}$$



**Lemma 2.** *Let  $\beta, M, N \geq 1$ . Taken  $r$  such that for some  $n$  we have that*

$$\|q_n \alpha\| < M \|q_n \alpha\|^{1/\beta} \leq 2r \leq \frac{1}{N} \|q_{n-1} \alpha\| < \|q_{n-1} \alpha\|$$

*then*

$$\mu\{x : \tau_r^\alpha(x) < (2r)^{-\beta}\} \leq \frac{1}{N} + \frac{1}{M^\beta}.$$

*Proof.* From hypothesis

$$(2r)^{-\beta} \leq \frac{1}{M^\beta \|q_n \alpha\|},$$

thus

$$\{x : \tau_r^\alpha(x) < (2r)^{-\beta}\} \subset \{x : \tau_r^\alpha(x) < \frac{1}{M^\beta \|q_n \alpha\|}\}.$$

Last set can be thought as a finite union and its measure given by

$$\sum_{k \in \mathbb{N}, 1 \leq k \leq q_n} \mu\{x : \tau_r^\alpha(x) = k\} + \sum_{k \in \mathbb{N}, q_n < k < \frac{1}{M^\beta \|q_n \alpha\|}} \mu\{x : \tau_r^\alpha(x) = k\}$$

so Proposition 6 and Proposition 4 imply:

$$\begin{aligned} \mu\{\tau_r^\alpha < \frac{1}{M^\beta \|q_n \alpha\|}\} &\leq 2r \cdot q_n + \|q_n \alpha\| \cdot \frac{1}{M^\beta \|q_n \alpha\|} \leq \frac{1}{N} \|q_{n-1} \alpha\| q_n + \frac{1}{M^\beta} \leq \\ &\leq \frac{1}{N} + \frac{1}{M^\beta}. \end{aligned}$$

□

*Proof.* (of Theorem 3) We denote by  $\mu^2$  the Lebesgue measure on  $\mathbb{T}^2$  and by  $\mu$  the Lebesgue measure on  $\mathbb{S}^1$ .

Since we consider a ratio of logarithms, the  $\liminf$  is preserved by considering the limit along the sequence  $r_i := e^{-i}$  (see lemma 1).

Take  $\beta < \gamma$ . Our interest lies in this sequence of subsets of the two-torus:

$$\overline{A}_i := \{x \in \mathbb{T}^2 : \tau_{r_i}^{(\alpha, \alpha')}(x) < (2r_i)^{-\beta}\} = \{x \in \mathbb{T}^2 : \frac{\log \tau_{r_i}^{(\alpha, \alpha')}(x)}{-\log r_i} + \frac{\beta \log 2}{-\log r_i} < \beta\}$$

if we prove that the measures of  $\overline{A}_i$  are summable, Borel-Cantelli lemma will imply that the measure of their set-theoretic  $\limsup$  (set of points that fall infinitely often in the sequence) is zero. Thus we have

$$\mu^2\{\liminf_i \frac{\log \tau_{r_i}^{(\alpha, \alpha')}}{-\log r_i} < \beta\} \leq \mu^2(\limsup_i \{\frac{\log \tau_{r_i}^{(\alpha, \alpha')}}{-\log r_i} < \beta\}) = 0$$

The thesis follows taking  $\beta$  arbitrarily near to  $\gamma$ .

We reduce the two dimensional problem to a one-dimensional one by defining the following subsets of the circle

$$A_i := \{x \in \mathbb{S}^1 : \tau_{r_i}^\alpha(x) < (2r_i)^{-\beta}\}$$

$$A'_i := \{x \in \mathbb{S}^1 : \tau_{r_i}^{\alpha'}(x) < (2r_i)^{-\beta}\}$$

and observing that  $\mu^2(\overline{A}_i) \leq \min(\mu(A_i), \mu(A'_i))$ .

To prove that  $\overline{A}_i$  are summable we will sometimes bound its measure with  $\mu(A_i)$ , sometimes with  $\mu(A'_i)$ . In fact, we will prove that two appropriate subsequences

of  $A_i$  and  $A'_i$  are summable and that the union of the indexes of the subsequences covers a neighbourhood of infinity.

The sets of indexes we take are, respectively,  $\bigcup_n I_n$  and  $\bigcup_n I'_n$  where  $I_n$  and  $I'_n$  are used to group subsets of consecutive indexes:

$$I_n := \{i > 0 : M_n \|q_n \alpha\|^{\frac{1}{\beta}} \leq 2r_i \leq \frac{1}{N_n} \|q_{n-1} \alpha\|\},$$

$$I'_n := \{i > 0 : M'_n \|q'_n \alpha'\|^{\frac{1}{\beta}} \leq 2r_i \leq \frac{1}{N'_n} \|q'_{n-1} \alpha'\|\};$$

this particular choice of  $M_n, M'_n, N_n, N'_n$  will serve our purposes

$$(4.1) \quad M_n = n, M'_n = n \quad N_n = \frac{1}{M'_{n-1}} \frac{\|q_{n-1} \alpha\|}{\|q'_{n-1} \alpha'\|^{\frac{1}{\beta}}} \quad N'_n = \frac{1}{M_n} \frac{\|q'_{n-1} \alpha'\|}{\|q_n \alpha\|^{\frac{1}{\beta}}}.$$

If the two sequences of intervals

$$(4.2) \quad [M_n \|q_n \alpha\|, \frac{1}{N_n} \|q_{n-1} \alpha\|], \quad [M'_n \|q'_n \alpha'\|^{\frac{1}{\beta}}, \frac{1}{N'_n} \|q'_{n-1} \alpha'\|]$$

are eventually non-empty then they cover a neighbourhood of zero from the right (because they alternate and equation (4.1) forces them to have equal extremes or overlap), thus  $\bigcup_n I_n \cup I'_n$  covers a neighbourhood of infinity.

Now we use the fact that  $(\alpha, \alpha') \in Y_\gamma$  and Proposition 4 to show that the ratio between the extremes of the intervals in (4.2) eventually grows to infinity (being in particular bigger than 1 and then forcing intervals to be not empty):

$$\frac{\frac{1}{N_n} \|q_{n-1} \alpha\|}{M_n \|q_n \alpha\|^{\frac{1}{\beta}}} = \frac{M'_{n-1} \|q'_{n-1} \alpha'\|^{\frac{1}{\beta}}}{M_n \|q_n \alpha\|^{\frac{1}{\beta}}} \geq \frac{n-1}{n} \left( \frac{q_{n+1}}{2q'_n} \right)^{\frac{1}{\beta}} \geq \frac{n-1}{n} \left( \frac{q_n^\gamma}{2q'_n} \right)^{\frac{1}{\beta}} \rightarrow \infty$$

$$\frac{\frac{1}{N'_n} \|q'_{n-1} \alpha'\|}{M'_n \|q'_n \alpha'\|^{\frac{1}{\beta}}} = \frac{M_n \|q_n \alpha\|^{\frac{1}{\beta}}}{M'_n \|q'_n \alpha'\|^{\frac{1}{\beta}}} \geq \left( \frac{q'_{n+1}}{2q_{n+1}} \right)^{\frac{1}{\beta}} \geq \left( \frac{q_{n+1}^\gamma}{2q_{n+1}} \right)^{\frac{1}{\beta}} \rightarrow \infty$$

To prove summability we need a trick to apply Lemma 2 to the whole bunch of  $r_i$  contained in a single set  $I_n$ . This is based on a simple remark: if  $2r \in [M \|q_n \alpha\|^{\frac{1}{\beta}}, \frac{1}{N} \|q_{n-1} \alpha\|]$  then  $2r \cdot c \in [cM \|q_n \alpha\|^{\frac{1}{\beta}}, \frac{1}{N/c} \|q_{n-1} \alpha\|]$ .

Let  $\ell_n := \lceil \log \frac{\frac{1}{N_n} \|q_{n-1} \alpha\|}{M_n \|q_n \alpha\|^{\frac{1}{\beta}}} \rceil$  be a bound to the cardinality of  $I_n$ . We apply  $\ell_n$  times lemma 2 with  $M = M_n, M_n \cdot e, \dots, M_n \cdot e^{\ell_n - 1}$  and  $N = N_n \cdot e^{\ell_n - 1}, N_n \cdot e^{\ell_n - 2}, \dots, N_n$ :

$$\begin{aligned} \sum_{i \in I_n} \mu(A_i) &\leq \frac{1}{N_n} \left( \frac{1}{e^{\ell_n - 1}} + \frac{1}{e^{\ell_n - 2}} + \dots + 1 \right) + \frac{1}{M_n^\gamma} \left( 1 + \frac{1}{e^\gamma} + \dots + \frac{1}{e^{\gamma(\ell_n - 1)}} \right) \\ &\leq \frac{1}{N_n} \left( \frac{1}{1 - e^{-1}} \right) + \frac{1}{M_n^\gamma} \left( \frac{1}{1 - e^{-\gamma}} \right). \end{aligned}$$

This argument applies equivalently to primed sequence.  $\gamma > 1$  so  $\frac{1}{M_n^\gamma}, \frac{1}{M_n^{\gamma\gamma}}$  are summable. Last step is to show summability of  $\frac{1}{N_n}, \frac{1}{N'_n}$ :

$$N_n = \frac{\|q_{n-1} \alpha\|}{M'_{n-1} \|q'_{n-1} \alpha'\|^{\frac{1}{\beta}}} \geq \frac{1}{n-1} \frac{q'_n{}^{\frac{1}{\beta}}}{2q_n} \geq \frac{1}{2(n-1)} q_n^{\frac{\gamma}{\beta} - 1}$$

$$N'_n = \frac{\|q'_{n-1} \alpha'\|}{M_n \|q_n \alpha\|^{\frac{1}{\beta}}} \geq \frac{1}{n} \frac{q_{n+1}^{\frac{1}{\beta}}}{2q'_n} \geq \frac{1}{2n} q'_n{}^{\frac{\gamma}{\beta} - 1}$$

But  $\gamma/\beta - 1 > 0$  and by Proposition 3 both  $q_n, q'_n$  eventually grow faster than any power of  $n$  so  $N_n, N'_n \geq n^\delta$  for some  $\delta > 1$ .  $\square$

**Remark 2.** A statement similar to theorem 3 follows from a result in a paper by Bugeaud and Chevallier ([6]). From their Theorem 2 it follows that for every function  $\phi : \mathbb{N} \rightarrow \mathbb{R}^+$  tending to 0, there exist rationally independent  $\bar{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$  such that the following subset of  $\mathbb{R}^2$  has zero Lebesgue measure:

$$\mathcal{W}_\phi(\bar{\alpha}) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \max_{i=1,2} \{ \|n\alpha_i - x_i\| \} < \phi(|n|) \text{ holds for infinitely many } n \in \mathbb{Z} \right\}.$$

With an appropriate choice of  $\phi$  and passing to the complement, this implies that for a fixed  $\gamma$  there exists a vector  $\bar{\alpha}$  such that the following subset of  $\mathbb{T}^2$  has full Lebesgue measure:

$$\left\{ \bar{x} \in \mathbb{T}^2 : \frac{-\log \text{dist}(T_{\bar{\alpha}}^n(\bar{x}), (0,0))}{\log n} \leq \frac{1}{\gamma} \text{ definitively for } n \in \mathbb{N} \right\}$$

By Propositions 11 and 13 this is equivalent to the existence of a translation on the torus such that  $\underline{R}(\bar{x}, (0,0)) \geq \gamma$  for Lebesgue almost every  $\bar{x} \in \mathbb{T}^2$ .

## 5. REPARAMETRIZATION OF FLOWS AND HITTING TIME

In this section we show that the time one map of the mixing flow on the 3-torus constructed by Fayad (see Theorem 4) has properties similar to the ones described in Theorem 3. Fayad's example is a flow, hence we also consider the concept of hitting time in the context of continuous time dynamical systems:

**Definition 5.** Let  $\Phi_t : X \rightarrow X$  be a flow on  $X$ . Take  $x \in X$  and consider the time needed for  $x$  to enter in a ball  $B_r(x_0)$ :

$$\tau_r^{(\Phi_t)}(x, x_0) := \inf\{t > 0 \mid \Phi_t(x) \in B_r(x_0)\}.$$

Working as in (2.1), this definition naturally provides a definition of hitting time indicator for continous systems and which we will denote as  $R^{(\Phi_t)}$  (while the indicator of the discrete system given by a map  $T$  will be denoted by  $R^{(T)}$ ).

We now consider the relations between hitting time in flows and in associated discrete time systems, as time-1 maps or Poincaré sections. As a first trivial remark we note that the hitting time indicator of a flow and its time-1 map can be quite different, hence we have to be careful to specify when discrete or continuous time is considered. In fact, think of a rotation on the circle by an irrational number  $\alpha$  of type 1 and consider the associated flow given by  $\Phi_t(x) = x + t\alpha \pmod{1}$  which is such that  $T_\alpha = \Phi_1$ . In this case we have that for each fixed  $x_0$

$$R^{(T_\alpha)}(x, x_0) = 1 \quad \text{while} \quad R^{(\Phi_t)}(x, x_0) = 0$$

for almost every  $x$ . In general however we have the following immediate relation between the flow and the time 1 associated map

**Proposition 7** (Flow and time 1 map). *If  $\Phi_t$  is a flow, as above, we have that if  $r$  is small enough*

$$\tau_r^{(\Phi_1)}(x, y) \geq \tau_r^{(\Phi_t)}(x, y)$$

for every  $x, y$  and hence

$$\overline{R}^{(\Phi_1)}(x, y) \geq \overline{R}^{(\Phi_t)}(x, y) \text{ and } \underline{R}^{(\Phi_1)}(x, y) \geq \underline{R}^{(\Phi_t)}(x, y).$$

Now we want to consider the case of a Poincaré section of a translation on the  $d$ -torus  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ . We introduce some general notation which will be used in the following. We equip  $\mathbb{T}^d$  with the sup distance: if  $x = (x_1, \dots, x_d)$  and  $y = (y_1, \dots, y_d)$  then  $\text{dist}(x, y) = \sup_i(|x_i - y_i|)$ . A translation by a vector  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$  on  $\mathbb{T}^d$  is the function  $T_\alpha : \mathbb{T}^d \rightarrow \mathbb{T}^d$  given by

$$(x_1, \dots, x_n) \mapsto (x_1 + \alpha_1, \dots, x_n + \alpha_n).$$

The translation is said to be irrational if the numbers  $1, \alpha_1, \dots, \alpha_d$  are rationally independent. In this case  $T_\alpha$  is uniquely ergodic and its invariant measure is the Lebesgue one.

Now let us consider the translation flow  $\Phi_t$  on  $\mathbb{T}^d$  with direction  $\alpha \in \mathbb{R}^d$ . This is the flow generated integrating the constant vector field  $X(x) = \alpha$ . As above, the flow is said to be irrational if  $\alpha_1, \dots, \alpha_d$  are rationally independent. Also in this case  $\Phi_t$  is uniquely ergodic, and the invariant measure is the Lebesgue one ([10]).

Let  $\Phi_t$  be a translation flow on  $\mathbb{T}^d$  and let  $\mathbb{T}^{d-1}$  be the torus of codimension 1 given by  $x_d = c$  (we will consider on  $\mathbb{T}^{d-1}$  the metric induced by the inclusion in  $\mathbb{T}^d$ ). Let us consider the map  $T_{\mathbb{T}^{d-1}}^\Phi : \mathbb{T}^{d-1} \rightarrow \mathbb{T}^{d-1}$  which is the Poincaré section of  $\Phi_t$  on  $\mathbb{T}^{d-1}$ . Let  $\pi(x) = \Phi_{t_0}(x)$  with  $t_0 = \inf\{t > 0 \mid \Phi_t(x) \in \mathbb{T}^{d-1}\}$  be the "projection" on  $\mathbb{T}^{d-1}$  induced by  $\Phi_t$ . Then the following relation holds

**Proposition 8** (Flow and a section). *Let  $T_{\mathbb{T}^{d-1}}^\Phi$  as above and  $y \in \mathbb{T}^{d-1}$ , let  $x \in \mathbb{T}^d$ . Then there are constants  $K, C > 0$  which depends also on the parametrization speed and the angle between flow and section, such that*

$$\tau_r^{(\Phi_t)}(x, y) \geq C \tau_{Kr}^{(T_{\mathbb{T}^{d-1}}^\Phi)}(\pi(x), y)$$

for every  $x$  such that  $\pi(x) \neq y$ , when  $r$  is small enough. Then under these assumptions

$$\overline{R}^{(\Phi_1)}(x, y) \geq \overline{R}^{(T_{\mathbb{T}^{d-1}}^\Phi)}(\pi(x), y) \text{ and } \underline{R}^{(\Phi_1)}(x, y) \geq \underline{R}^{(T_{\mathbb{T}^{d-1}}^\Phi)}(\pi(x), y).$$

We remark that if a flow is irrational then also its Poincaré section, as defined above is irrational.

Now we study invariance of continous hitting time indicator with respect to a wide class of time reparametrizations. Given a vector  $\alpha$  and a strictly positive, smooth function  $\phi : \mathbb{T}^d \rightarrow \mathbb{R}$  we can define the reparametrized translation flow  $\Phi_t^{\phi, \alpha}$  with velocity  $\phi$ , as the flow given by the vector field  $\phi(x)\alpha$ , that is by integrating the system

$$\dot{x} = \phi(x)\alpha.$$

The new flow has the same orbits as the original one and preserves a measure which is absolutely continuous with respect to the Lebesgue measure having density  $\frac{1}{\phi(x)}$ . Moreover if the original flow is ergodic then also the reparametrized one is. Under quite general assumptions a regular reparametrizations  $\phi$ , does not change the hitting time behavior of the flow.

**Proposition 9** (Reparametrization of a flow). *If  $\phi$  is  $C^1$  and strictly positive on the (compact) torus  $\mathbb{T}^d$  as above then there exists a constant  $C \geq 1$  such that  $\frac{1}{C} < \phi(x) < C$  for all  $x \in \mathbb{T}^d$ . Thus*

$$C \tau_r^{(\Phi_t)}(x, y) \geq \tau_r^{(\Phi_t^{\phi, \alpha})}(x, y) \geq \frac{1}{C} \tau_r^{(\Phi_t)}(x, y);$$

this implies

$$R^{(\Phi_t)}(x, y) = R^{(\Phi_t^{\phi, \alpha})}(x, y).$$

Combining the above propositions we obtain a way to estimate from below the hitting time indicators of time-1 maps of reparametrizations of rotations (or even other kind of maps constructed in a similar way) by the hitting time of discrete time translations with the same angles.

**Proposition 10.** *Let  $\Phi = \Phi_1^{(\phi, \alpha)}$  be the time one map of a bounded speed reparametrization of a translation flow on the direction vector  $\alpha = (1, \alpha, \alpha')$  and  $T = T_{\mathbb{T}^2}^{(\alpha, \alpha')}$  be the translation on the two torus given by rationally independent angles  $(\alpha, \alpha')$ , then for any  $y \in \mathbb{T}^3$ ,  $b \in \mathbb{T}^2$*

$$\overline{R}^\Phi(x, y) \geq \overline{R}^T(a, b)$$

$$\underline{R}^\Phi(x, y) \geq \underline{R}^T(a, b)$$

for almost each  $(a, x)$  (with respect to the Lesbegue measure in the product  $\mathbb{T}^2 \times \mathbb{T}^3$ ).

*Proof.* Since  $T_{\mathbb{T}^2}^{(\alpha, \alpha')}$  is ergodic, and  $\mu(b) = 0$ , then  $\overline{R}^T(a, b)$  is almost everywhere constant when  $a$  varies. This constant moreover is obviously the same for each  $b$  (see proposition 1). We will show that  $\overline{R}^\Phi(x, y)$  is a.e. greater than this constant when  $x$  varies. The same can be done for  $\underline{R}^\Phi(x, y)$ .

By the assumption,  $\Phi_1^{(\phi, \alpha)}$  is a time 1 map of a flow on  $\mathbb{T}^3$  which is the reparametrization of a translation flow  $\Phi_t$  with direction  $\alpha = (1, \alpha, \alpha')$ .

If  $y = (y_1, y_2, y_3) \in \mathbb{T}^3$ , we can consider the Poincaré section  $T_{\mathbb{T}^2}^{(\alpha, \alpha')}$  of this flow on a 2-torus  $\mathbb{T}^2$  which is the set of points in  $\mathbb{T}^3$  whose first coordinate is  $y_1$  (this torus hence contains  $y$ ). Then, combining Propositions 7, 8 and 9 there is a constant  $K, C > 0$  such that for any  $x, y \in \mathbb{T}^3$ , when  $r$  is small

$$\tau_r^{\Phi_1^{(\phi, \alpha)}}(x, y) \geq C \tau_{Kr}^{T_{\mathbb{T}^2}^{(\alpha, \alpha')}}(\pi(x), y)$$

where  $\pi(x)$  is the projection defined before Proposition 8. The statement follows, because  $\pi^{-1}(A - \{y\})$  is a full measure subset of  $\mathbb{T}^3$  when  $A$  is a full measure set in  $\mathbb{T}^2$ .  $\square$

In Proposition 9 we have seen that reparametrizing a flow, does not change the hitting time indicators. On the other side a flow reparametrizations can have surprisingly different ergodic properties compared to the original flow. A reparametrization of a translation can be mixing as we will see below.

Let  $Y$  be the set of couples  $(\alpha_1, \alpha_2) \in \mathbb{R}^2 - \mathbb{Q}^2$  such that the respective best approximant denominators  $q_n, q'_n$  satisfies definitively

$$q'_n \geq e^{3q_n}$$

$$q_{n+1} \geq e^{3q'_n}.$$

Note that  $Y$  is an uncountable and dense set and  $Y \subset Y_\infty$ .

The main result of [10] states:

**Theorem 4.** *(Fayad's example) There is a strictly positive analytic function  $\phi$  on  $\mathbb{T}^3$ , such that for any  $(\alpha_1, \alpha_2) \in Y$  the reparametrization with speed  $\frac{1}{\phi}$  of the irrational flow given by the vector  $(\alpha_1, \alpha_2, 1)$  is mixing.*

We remark that the particular choice of numbers  $\alpha_1$  and  $\alpha_2$  given by the above equations is important. It is proved ([10], Thm. 2) that for a generic choice of  $\alpha_1$  and  $\alpha_2$  we cannot have a similar result. By Proposition 10 and Theorem 3 the time-1 map of this system has infinite hitting time indicator. This will provide an example of a smooth mixing system with no relation between hitting time and measure (in other words a smooth, mixing system with no logarithm law).

**Corollary 1.** *Let  $(\mathbb{T}^3, \Phi_1^{\phi, \alpha})$  be the time 1 map of the system described in Theorem 4 equipped with its natural absolutely continuous invariant measure  $\mu$ . This system is mixing and we have that for each  $x$ ,  $d_\mu(x) = 3$ , while*

$$\overline{R}(y, x) = \underline{R}(y, x) = \infty$$

for almost each  $y$ .

*Proof.* This map is the time-1 map of a strictly positive speed reparametrization of a translation flow  $(\mathbb{T}^3, \Phi_t)$  with angle  $\alpha = (1, \alpha, \alpha')$  and  $(\alpha, \alpha') \in Y \subset Y_\infty$ . By proposition 10 its hitting time indicator is greater or equal than the one given by the translation  $T_{\mathbb{T}^2}^{(\alpha, \alpha')}$  on  $\mathbb{T}^2$  given by the angles  $(\alpha, \alpha')$  and this, by Theorem 3 are infinite.  $\square$

## 6. DECAY OF CORRELATIONS AND HITTING TIME

It is well known that many kinds of chaotic dynamics "mixes" the phase space. Decay of correlation speed gives a quantitative estimation for the speed of this mixing behavior. We recall the definition.

**Definition 6.** *A system  $(X, T, \mu)$ , where  $T$  is  $\mu$ -invariant, is said to have decay of correlations with power law speed with exponent  $p$  if there exists  $\Phi(n)$  such that, for all Lipschitz observables  $\phi, \psi : X \rightarrow \mathbb{R}$  on  $X$*

$$\left| \int \phi \circ T^n \psi d\mu - \int \phi d\mu \int \psi d\mu \right| \leq \|\phi\| \|\psi\| \Phi(n).$$

and  $\lim_{n \rightarrow \infty} \frac{-\log \Phi(n)}{\log n} = p$ . Here  $\|\cdot\|$  is the Lipschitz norm<sup>3</sup>.

We remark that by the above definition, decay of correlations slower than any negative power law (logarithmic decay for example) are considered as power laws with zero exponent.

**Lemma 3.** *Let  $A_n = T^{-n}(B_r(x_0))$ . If  $(X, T, \mu)$  is a system satisfying definition 6 then for each small  $\lambda > 0$*

$$(6.1) \quad \mu(A_k \cap A_j) \leq \mu(B_{(\lambda+1)r}(x_0))^2 + \frac{4\Phi(k-j)}{\lambda^2 r^2}$$

holds eventually as  $r \rightarrow 0$ .

*Proof.* Let  $\phi_\lambda$  be a Lipschitz function with norm less than  $\frac{2}{\lambda r}$  such that  $\phi_\lambda(x) = 1$  for all  $x \in B_r(x_0)$ ,  $\phi_\lambda(x) = 0$  if  $x \notin B_{(1+\lambda)r}(x_0)$ , the result follows directly by definition 6, by the remarking that by invariance of  $\mu$

$$(6.2) \quad \mu(A_k \cap A_j) \leq \int \phi_r \circ T^{k-j} \phi_r d\mu.$$

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<sup>3</sup> This is a definition for Lipschitz observables which is one of the weakest possible statements. In many systems decay of correlation is proved for observables in other functional spaces with weaker norms (Holder functions e.g.). In this cases the decay of correlation for Lipschitz observables follows by estimating the weaker norms by the Lipschitz one.

□

We now prove that in systems with at least polynomial decay of correlations and "good" measure the hitting time in small balls is related to the inverse of the ball measure up to an error which is controlled by decay of correlation speed.

**Theorem 5.** *Let  $(X, T, \mu)$  be a (probability) measure preseving transformation on a metric space. Suppose that for the point  $x_0$  we have  $\mu(\{x_0\}) = 0$ ,  $0 < \underline{d}_\mu(x_0) \leq \overline{d}_\mu(x_0) < \infty$  and the measure  $\mu$  satisfies the following property<sup>4</sup>: there exist  $C, \beta > 0$  such that for each  $r, \lambda > 0$  small enough*

$$(6.3) \quad \mu(B_{(1+\lambda)r}(x)) \leq \mu(B_r(x))(1 + C\lambda^\beta).$$

*If the system has polynomial decay of correlation with exponent  $p$ , then*

$$\limsup_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log \mu(B_r(x_0))} \leq 1 + \frac{2\overline{d}_\mu(x_0) + 2}{\underline{d}_\mu(x_0)p}$$

*holds for  $\mu$ -almost each  $x$ .*

*Proof.* (of Theorem C) Lower bound follows from Proposition 2. An absolutely continuous invariant measure on a manifold of dimension  $d$  having continuous and strictly positive density satisfies the hypotheses 6.3 on the measure at every point. In addition it is exact dimensional with  $\underline{d}_\mu = \overline{d}_\mu = d$ . Upper bound follows from Theorem 5 (cfr. also Remark 1). □

*Proof.* (of Theorem 5) Since we consider a ratio of logarithms and  $\overline{d}_\mu(x_0) < \infty$ , the measure  $\mu(B_{e^{-n}}(x_0))$  will decrease at most exponentially fast, and without loss of generality we can restrict to a sequence of radii  $r_n = e^{-n}$  (see Lemma 1).

Set  $\zeta = \frac{2\overline{d}_\mu(x_0)+2}{p} + \epsilon$ ,  $\epsilon > 0$  and

$$A_n = \bigcup_{1 \leq i \leq e^{\zeta n} \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor} T^{-i}(B_{r_n}(x_0)).$$

To prove our thesis it is sufficient to prove the following statement: for every  $\zeta > \frac{2\overline{d}_\mu(x_0)+2}{p}$ ,  $\mu$ -almost every  $x \in X$  belongs eventually to  $A_n$ . Indeed we have definitively  $\tau_{r_n}(x, x_0) \leq e^{\zeta n} \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor$ , then

$$\frac{\log(\tau_{r_n}(x, x_0))}{-\log \mu(B_{r_n}(x_0))} \leq \frac{\zeta n + \log \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor}{-\log \mu(B_{r_n}(x_0))} \leq \frac{2\overline{d}_\mu(x_0) + 2 + \epsilon p}{(\underline{d}_\mu(x_0) - \epsilon)p} + \frac{\log \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor}{\log \mu(B_{r_n}(x_0))^{-1}}.$$

And the thesis follows by letting  $n \rightarrow \infty$  and  $\epsilon \rightarrow 0$ .

Hence, we set  $m_n = e^{\zeta n} \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor$  and consider

$$Z_n(x) = \sum_{i=1}^{m_n} 1_{T^{-i}(B_{r_n}(x_0))}(x).$$

If  $Z_n(x) > 0$  then  $x \in A_n$ . If we prove that  $\frac{Z_n(x)}{\mathbf{E}(Z_n)} \rightarrow 1$  almost everywhere for every  $\zeta > \frac{2\overline{d}_\mu(x_0)+2}{p}$  then the statement is proved.

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<sup>4</sup> Which roughly means that the measure goes polynomially to zero for every annulus in a neighbourhood of  $x_0$ ; this is due to the technical requirement that we assume decay of correlations of observables but we want to use it for events (see Lemma 3).

The idea is to estimate  $\mathbf{E}((Z_n)^2)$  and find an upper bound which ensures that the distribution of the possible values of  $Z_n$  is not too far from the average  $\mathbf{E}(Z_n)$ . To compare  $Z_n$  with its average  $\mathbf{E}(Z_n)$  we consider

$$Y_n = \frac{Z_n}{\mathbf{E}(Z_n)} - 1 = \frac{Z_n - \mathbf{E}(Z_n)}{\mathbf{E}(Z_n)}.$$

When  $Y_n = 0$ ,  $Z_n = \mathbf{E}(Z_n)$ , thus we need to prove that  $Y_n \rightarrow 0$  almost everywhere.

We denote the preimages of the balls as  $B_k = T^{-k}(B_{r_n}(x_0))$ .

We have that

$$(6.4) \quad \mathbf{E}((Z_n)^2) = \sum_{k=1}^{m_n} \mu(B_k) + 2 \sum_{k,j \leq m_n, k > j} \mu(B_k \cap B_j).$$

The second summand on the right side of (6.4) can be thought as the sum of the off-diagonal elements of a matrix with entries  $\mu(B_k \cap B_j)$ . We split this sum in two parts by considering separately the entries ‘near’ or ‘far’ the diagonal. We measure this ‘nearness’ in the following way: take  $\alpha$  such that  $\frac{2\bar{d}_\mu(x)+2}{p} < \alpha < \zeta$  and split the sum as

$$(6.5) \quad \begin{aligned} & \sum_{k,j \leq m_n, k > j} \mu(B_k \cap B_j) = \\ & = \sum_{k,j \leq m_n, k > j, k < j + e^{\alpha n}} \mu(B_k \cap B_j) + \sum_{k,j \leq m_n, k \geq j + e^{\alpha n}} \mu(B_k \cap B_j). \end{aligned}$$

Since  $\mu(B_k \cap B_j) \leq \mu(B_k)$ , the first summation in (6.5) can be largely estimated as follows:

$$(6.6) \quad \sum_{k,j \leq m_n, k > j, k < j + e^{\alpha n}} \mu(B_k \cap B_j) \leq e^{\alpha n} \mathbf{E}(Z_n).$$

To estimate the second summation we use hypothesis on speed of decay of correlations. From Lemma 3, taking  $\lambda = n^{-\frac{2}{\beta}}$  (the value of  $\beta$  depends on  $\mu$  as given in (6.3)) we have that

$$(6.7) \quad \mu(B_k \cap B_j) \leq \mu(B_{(1+n^{-\frac{2}{\beta}})r_n}(x_0))^2 + \frac{4\Phi(k-j)}{(n^{-\frac{2}{\beta}})^2 r_n^2}.$$

with  $\Phi$  as in definition 6 is polynomially decaying with exponent  $p$ . By equation (6.3) we have that

$$(6.8) \quad \mu(B_{(1+n^{-\frac{2}{\beta}})r_n}(x_0))^2 \leq \mu(B_{r_n}(x_0))^2 (1 + Cn^{-2})^2.$$

Using the bounds in (6.7) and (6.8) we obtain

$$\begin{aligned} & \sum_{k,j \leq m_n, k \geq j + e^{\alpha n}} \mu(B_k \cap B_j) \leq \\ & \leq \sum_{k,j \leq m_n, k \geq j + e^{\alpha n}} \left[ \mu(B_{(1+n^{-\frac{2}{\beta}})r_n}(x_0))^2 + \frac{4\Phi(k-j)}{(n^{-\frac{2}{\beta}})^2 r_n^2} \right] \leq \\ & \leq \frac{1}{2} \mathbf{E}((Z_n))^2 (1 + Cn^{-2})^2 + \frac{(m_n)^2}{(n^{-\frac{2}{\beta}})^2 (r_n)^2} 4\Phi(e^{\alpha n}). \end{aligned}$$



Hence by (6.6)

$$\begin{aligned}
 & \sum_{k,j \leq m_n, k > j} \mu(B_k \cap B_j) \leq \\
 (6.9) \quad & \leq e^{\alpha n} \mathbf{E}(Z_n) + \frac{1}{2} (\mathbf{E}(Z_n))^2 (1 + Cn^{-2})^2 + \frac{4(m_n)^2}{(n^{-\frac{2}{\beta}})^2 (r_n)^2} \Phi(e^{\alpha n})
 \end{aligned}$$

Now, plugging in (6.6) and (6.9) into (6.4), we obtain

$$\begin{aligned}
 \mathbf{E}((Z_n - \mathbf{E}(Z_n))^2) &= \mathbf{E}((Z_n)^2) - (\mathbf{E}(Z_n))^2 \leq \\
 &\leq (2e^{\alpha n} + 1) \mathbf{E}(Z_n) + (\mathbf{E}(Z_n))^2 (C^2 n^{-4} + 2Cn^{-2}) + \\
 &\quad + \frac{4(m_n)^2}{(n^{-\frac{2}{\beta}})^2 (r_n)^2} \Phi(e^{\alpha n}).
 \end{aligned}$$

which, in terms of  $Y_n$ , amounts to

$$\begin{aligned}
 \mathbf{E}((Y_n)^2) &\leq \frac{(e^{\alpha n+1} + 1) \mathbf{E}(Z_n) + (\mathbf{E}(Z_n))^2 (C^2 n^{-4} + 2Cn^{-2})}{(\mathbf{E}(Z_n))^2} + \\
 &\quad + \frac{4(m_n)^2}{(\mathbf{E}(Z_n))^2 (n^{-\frac{2}{\beta}})^2 (r_n)^2} \Phi(e^{\alpha n})
 \end{aligned}$$

and, since  $\mathbf{E}(Z_n) = m_n \cdot \mu(B_{r_n}(x_0)) = e^{\zeta n} \lfloor \mu(B_{r_n}(x_0))^{-1} \rfloor \mu(B_{r_n}(x_0))$ , we have

$$\mathbf{E}((Y_n)^2) \leq \frac{(e^{\alpha n+1} + 1)}{m_n \mu(B_{r_n}(x_0))} + \frac{4\Phi(e^{\alpha n})}{\mu(B_{r_n}(x_0))^2 (n^{-\frac{2}{\beta}})^2 (r_n)^2} + C^2 n^{-4} + 2Cn^{-2}$$

Finally, we note that  $\zeta > \alpha$  and  $p\alpha > 2\bar{d}_\mu(x) + 2$ , thus  $\mathbf{E}((Y_n)^2)$  goes to zero in a summable<sup>5</sup> way. This proves that  $\frac{Z_n}{\mathbf{E}(Z_n)} \rightarrow 1$  almost everywhere.  $\square$

**Corollary 2.** *Under the assumptions of the previous theorem, if  $T$  power law decay of correlations then*

$$\overline{R}(x, x_0) < \infty, \underline{R}(x, x_0) < \infty$$

$\mu$  for a.e.  $x \in X$ .

This gives us that no reparametrizations of translations with infinite hitting time indicator (as in Fayad's example, as showed in section 4 and 5) can be polynomially mixing, moreover.

**Corollary 3.** *If  $(X, T)$  is a map on an  $d$  dimensional manifold, having an absolutely continuous invariant measure with strictly positive density and  $\overline{R}(x, y) = R$  a.e., then the speed of decay of correlations of the system is a power law with exponent  $p \leq \frac{2d+2}{R-d}$ .*

*Proof.* (of Theorem D) Follows from Proposition 9, Proposition 5 and Corollary 3.  $\square$

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<sup>5</sup>We remark that  $\frac{\Phi(e^{\alpha n})}{\mu(B(x, r_n))^2 (n^{-\frac{2}{\beta}})^2 (r_n)^2} \leq C' n^{4/\beta} \frac{e^{-p\alpha n}}{e^{-n(2d+2)}} = n^{4/\beta} e^{-n[p\alpha - (2d+2)]}$ .

## 7. APPENDIX

In this section we recall and precise some results on the equivalence between different approaches to the hitting time problem.

Let us denote by  $\text{dist}(\cdot, \cdot)$  the distance on  $X$  and define  $d_n(x, y) = \min_{1 \leq i \leq n} \text{dist}(T^i(x), y)$ .

**Proposition 11.** *Given a system  $T$  on a metric space  $(X, \text{dist})$ :*

$$\underline{R}(x, x_0) = \left( \limsup_{n \rightarrow \infty} \frac{-\log d_n(x, x_0)}{\log n} \right)^{-1}$$

and

$$\overline{R}(x, x_0) = \left( \liminf_{n \rightarrow \infty} \frac{-\log d_n(x, x_0)}{\log n} \right)^{-1}.$$

*Proof.* Note that  $d_n \leq r$  if and only if  $\tau_r \leq n$ . Suppose  $\limsup_{n \rightarrow \infty} \frac{-\log d_n}{\log n} = a$  and take  $\epsilon > 0$ .

There exist infinitely many  $n$  such that  $\frac{-\log d_n}{\log n} \geq a - \epsilon$  that is  $d_n \leq n^{-a+\epsilon}$ , thus  $\tau_{n^{-a+\epsilon}} \leq n$ . Put  $r = n^{-a+\epsilon}$ ,  $n = r^{\frac{-1}{a-\epsilon}}$  and you find a sequence of radii going to zero such that  $\tau_r \leq r^{\frac{-1}{a-\epsilon}}$ , therefore  $\liminf_{r \rightarrow 0} \frac{\log \tau_r}{-\log r} \leq \frac{1}{a}$ .

Eventually  $\frac{-\log d_n}{\log n} \leq a + \epsilon$  or  $d_n \geq n^{-a-\epsilon}$ , thus  $\tau_{n^{-a-\epsilon}} \geq n$ . Putting  $r = n^{-a-\epsilon}$ ,  $n = r^{\frac{-1}{a+\epsilon}}$  we obtain a sequence of radii (decreasing slower than exponentially) for which  $\tau_r \geq r^{\frac{-1}{a+\epsilon}}$ , therefore  $\liminf_{r \rightarrow 0} \frac{\log \tau_r}{-\log r} \geq \frac{1}{a}$ .

This establishes a bijection ( $a \mapsto \frac{1}{a}$ ) between the non-negative quantities  $\underline{R}(x, x_0)$  and  $\limsup_{n \rightarrow \infty} \frac{-\log d_n}{\log n}$ . The other equation is treated similarly.  $\square$

**Proposition 12.** *For any function  $f : \mathbb{N} \rightarrow \mathbb{R}$*

$$\sup\{\beta : \liminf_n n^\beta f(n) = 0\} = \inf\{\beta : \liminf_n n^\beta f(n) = \infty\} = \limsup_n \frac{-\log f(n)}{\log n}$$

and

$$\sup\{\beta : \limsup_n n^\beta f(n) = 0\} = \inf\{\beta : \limsup_n n^\beta f(n) = \infty\} = \liminf_n \frac{-\log f(n)}{\log n}.$$

*Proof.* We will prove the first one, the other being similar. The equality between sup and inf is obvious. We will show:

$$\liminf_{n \rightarrow \infty} n^\beta f(n) = 0 \Rightarrow \limsup_{n \rightarrow \infty} \frac{-\log f(n)}{\log n} \geq \beta,$$

$$\liminf_{n \rightarrow \infty} n^\beta f(n) = \infty \Rightarrow \limsup_{n \rightarrow \infty} \frac{-\log f(n)}{\log n} \leq \beta.$$

Take  $\beta$  such that  $\liminf_n n^\beta f(n) = 0$  holds and  $\epsilon > 0$ . There exist infinitely many  $n$  such that

$$n^\beta f(n) < \epsilon \Leftrightarrow \beta \log n + \log f(n) < \log \epsilon \Leftrightarrow \frac{-\log f(n)}{\log n} > \beta - \frac{\log \epsilon}{\log n}$$

thus  $\limsup_n \frac{-\log f(n)}{\log n} \geq \beta$ .

Take  $\beta$  such that  $\liminf_n n^\beta f(n) = \infty$  holds. For every  $\epsilon > 0$  definitively

$$n^\beta f(n) > \frac{1}{\epsilon} \Leftrightarrow \beta \log n + \log f(n) > -\log \epsilon \Leftrightarrow \frac{-\log f(n)}{\log n} < \beta + \frac{\log \epsilon}{\log n}$$

thus  $\limsup_n \frac{-\log f(n)}{\log n} \leq \beta$ .  $\square$

**Proposition 13.**

$$(7.1) \quad \liminf_n n^\beta d_n(x, y) = 0 \Leftrightarrow \liminf_n n^\beta \text{dist}(T^n(x), y) = 0$$

thus

$$\limsup_n \frac{-\log d_n(x, y)}{\log n} = \limsup_n \frac{-\log \text{dist}(T^n(x), y)}{\log n}.$$

*Proof.* To prove double implication we first observe that  $0 \leq n^\beta d_n(x, y) \leq n^\beta \text{dist}(T^n(x), y)$ , which gives us one implication. Now take a subsequence  $\{n_k\}_{k \in \mathbb{N}}$  such that  $\lim_k n_k^\beta d_{n_k}(x, y) = 0$ . For each  $k$  we construct another subsequence  $\{n'_k\}_{k \in \mathbb{N}}$  taking  $n'_k$  the biggest integer such that  $d_{n'_k}(x, y) = \text{dist}(T^{n'_k}(x), y)$  and  $n'_k \leq n_k$ . We have that  $n'_k \rightarrow \infty$  and  $n'_k{}^\beta \text{dist}(T^{n'_k}(x), y) \leq n_k^\beta d_{n_k}(x, y) \rightarrow 0$ .

Second part of the statement follows for (7.1) and proposition 12.  $\square$

**Remark 3.** Concerning relations between the two approaches of waiting time and Borel-Cantelli we recall some results proved in [16]. A result like equation (1.2) implies that, for every  $x_0$  and almost every  $x$ ,

$$(7.2) \quad \lim_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log \mu(B_r(x_0))} = 1,$$

hence

$$(7.3) \quad \lim_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0)$$

when  $\mu(x_0) = 0$  and the local dimension exists.

Conversely there are systems for which  $\liminf_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0)$  but there are certain decreasing sequences of balls with  $s_n = \sum_{i=1}^n \mu(B_i) \rightarrow \infty$  such that  $\mu(\limsup_i T^{-i}(B_i)) = 0$ . This situation, however, cannot happen if we restrict to balls where  $s_n \sim n^\alpha$ ,  $\alpha > 0$ .

## REFERENCES

- [1] Abadi M., *Sharp error terms and necessary conditions for exponential hitting times in mixing processes*, Ann. Probab. **32** (2004), 243–264.
- [2] Athreya J.S., Margulis G. A., *Logarithm laws for unipotent flows, I*, preprint arXiv:0811.2806v2.
- [3] Barreira L., Saussol B., *Hausdorff dimension of measures via Poincaré recurrence*, Commun. Math. Phys. **219** (2001), 443–463.
- [4] Beresnevich V., Bernik V., Dodson M., Velani S., *Classical metric Diophantine approximation revisited*, preprint arXiv:0803.2351v1.
- [5] Boshernitzan M. D., *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.
- [6] Bugeaud Y., Chevallier N., *On simultaneous inhomogeneous Diophantine approximation*, Acta Arithm. **123** (2006).
- [7] Chernov N, Kleinbock D *Dynamical Borel-Cantelli lemmas for gibbs measures* Israel Journal of Mathematics **122** (2001), 1–27.
- [8] Choe G. H. and Seo B. K., *Recurrence speed of multiples of an irrational number*, Proc. Japan Acad. Ser. A **7** (2001), 134–7.
- [9] Dolgopyat D., *Limit theorems for partially hyperbolic systems*, Trans. Amer. Math. Soc. **356** (2004), 1637–1689.
- [10] Fayad B., *Analytic mixing reparametrizations of irrational flows*, Ergodic Theory and Dynamical Systems **22** (2002), 437–468.
- [11] Fayad B., *Mixing in the absence of the shrinking target property*, Bull. Lond. Math. Soc. **38** (2006), 829–838.

- [12] Fayad B., *Polynomial decay of correlations for a class of smooth flows on the two torus*, Bull. Soc. Math. France **129** (2001), 487–503.
- [13] Galatolo S., *Dimension and waiting time in rapidly mixing systems*, Math Res. Lett. **14**, No. 5-6, 797-805 (2007).
- [14] Galatolo S., *Dimension via waiting time and recurrence*, Math. Res. Lett. **12** (2005), 377–386.
- [15] Galatolo S., *Hitting time and dimension in axiom A systems, generic interval exchanges and an application to Birkoff sums*, J. Stat. Phys. **123** (2006), 111–124.
- [16] Galatolo S., Kim D. H., *The dynamical Borel-Cantelli lemma and the waiting time problems*, Indag. Math., vol. 18 (2007), no. 3, 421-434
- [17] Hill R., Velani S., *The ergodic theory of shrinking targets* Inv. Math. **119** (1995), 175–198.
- [18] Khintchin A., *Continued Fractions*, Univ. of Chicago Press (1964).
- [19] Kim D. H., *The shrinking target property of irrational rotations*, Nonlinearity **20** (2007), 1637–1643.
- [20] Kim D.H. , Marmi S. *The recurrence time for interval exchange maps*, Nonlinearity **21** (2008), 2201-2210 .
- [21] Kim D. H. and Seo B. K., *The waiting time for irrational rotations*, Nonlinearity **16** (2003), 1861–1868.
- [22] Kleinbock D. Y. , Margulis G. A., *Logarithm laws for flows on homogeneous spaces*. Inv. Math. **138** (1999), 451–494.
- [23] Kurzweil J., *On the metric theory of inhomogenous Diophantine approximations*, Studia Math. **15** (1955), 84–112.
- [24] Maucourant F., *Dynamical Borel Cantelli lemma for hyperbolic spaces*, Israel J. Math. **152** (2006), 143–155.
- [25] Masur H., *Logarithmic law for geodesics in moduli spaces*, Contemporary Mathematics **150** (1993), 229–245.
- [26] Pesin Y., *Dimension theory in dynamical systems*, Chicago lectures in Mathematics (1997).
- [27] Sondow J., *Irrationality measures, irrationality bases and a theorem of Jarnik*, preprint arXiv:math/0406300v1.
- [28] Sullivan D., *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics* Acta Mathematica **149** (1982), 215–237.
- [29] Tseng J., *On circle rotations and shrinking target property*, Discrete Contin. Dyn. Syst. **20** (2008) 1111-1122
- [30] Tseng J., *Three remarks on shrinking target properties*, preprint arXiv:0807.3298v2.
- [31] Yoccoz J.-C., *Petit diviseurs en dimension 1*, Astérisque **231** (1995), 89–242 (Appendix 1).

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